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ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series is designed to make material for the study of topics of special interest to students readily accessible in classroom quantity. Topics covered include natural numbers, positive integers, sets, well ordering, lower bound, upper bound, rational numbers, repeating decimals, real numbers, complete number systems, and irrational numbers. (MP)

SCHOOL MATHEMATICS STUDY GROUP

SP-28

SUPPLEMENTARY and ENRICHMENT SERIES

Order and the Real Numbers a Guided Tour

By Frank L. Wolf

2

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PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

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ORDER AND THE REAL NUMBERS -- A GUIDED TOUR

In view of the experience we have all had when the amount of money we have is less than the cost of an item we would like to buy, we are all familiar with the idea of one number being less than another. In this lesson and those that follow, we will be learning things about this important relationship between numbers. We start by considering the simplest kind of numbers -- the numbers 1, 2, 3, 4 and so forth -- the numbers we have used to count with since we were very young. We shall call these numbers the natural numbers. We could also call them the positive integers, but usually will not.

Instead of saying that one natural number is less than another, we may also say that the first number is smaller than the second and mean the same thing. Thus, 25 is less than 30 and, also, 25 is smaller than 30.

All of us are lazy and prefer a short way to write something rather than a long way to write it. It is convenient, therefore, to abbreviate "25 is less than 30" as "$25 < 30$". That is, we let "$<$" stand for the words "is less than".

We now present some problems for you. You are to decide in each case whether or not the statement is true or false. Record your answers to all the parts of problem 1. When you have finished, check your answers against those given at the back of the booklet.

1. (a) $4 < 7$
   (b) $12 < 10$
   (c) $193 < 200$
   (d) $3 < 5$
   (e) $10{,}000 < 9{,}999$
   (f) $47 + 6 < 47$
   (g) $133 < 133 + 941$
   (h) $5 + 2 < 5$
   (i) $14{,}397 < 14{,}397 + 1$
   (j) If x is any natural number, $x < x + 1$.
   (k) If x and y are natural numbers, then $x < x + y$.
   (ℓ) $5 < 7$ and $7 < 53$
   (m) $5 < 7$ and $7 < 6$
   (n) $391 < 400$ and $400 < 1940$
   (o) $391 < 391$

It is pleasant to have more money than the cost of something you want to buy, and it is convenient to have an abbreviation for the phrase "is more than". The symbol we use for this abbreviation is ">". It is true, for example, that $12 > 10$.

If you find yourself getting confused between "<" and ">", remember that the shape of the symbol is meant to be helpful. The "large", open end of the symbol is next to the name of the number which, it is claimed, is the large number. The small point of the symbol points towards the name of the one claimed to be smaller.

True or False -- as before:

2. (a) $37 > 73$
   (b) $37 > 17$
   (c) $49341 > 49340$
   (d) If x is a natural number, $x + 1 > x$.
   (e) If x and y are natural numbers and $x < y$, then $y > x$.
   (f) $59 > 40$ and $40 > 32$
   (g) $73 > 58$ and $58 > 60$

The cost of a bicycle is usually between that of a candy bar and that of a car. 10 is between 5 and 17, because $5 < 10$ and $10 < 17$. Also 39 is between 1000 and 30 because $1000 > 39$ and $39 > 30$. In general, we say that a is between b and c if either $b < a$ and $a < c$, or $b > a$ and $a > c$.

True or False -- as before:

3. (a) 4 is between 2 and 10.
   (b) 4 is between 10 and 2.
   (c) 4 is between 7 and 10.
   (d) 37 is between 10 and 50.
   (e) 497 is between 400 and 500.
   (f) 497 is between 1 and 1000.
   (g) 394 is between 395 and 500.
   (h) 20 is between 10 and 16.

The natural numbers between 5 and 10 are the numbers 6, 7, 8, and 9. If we wish to talk about this collection of numbers, we will write "{6, 7, 8, 9}" to stand for the collection. We will usually refer to any

collection as a set. The things that are in a set are called members (or elements) of the set. The set {3, 4, 7} has the members 3, 4, and 7. That is, 3 is a member of this set and so are 4 and 7.

We read "{3, 4, 7}" as "the set whose members are 3, 4, and 7". The symbols "{" and "}" are called braces. Whenever we use braces we will be talking about sets. The left brace, "{", may be considered to be an abbreviation for "the set whose members are". The right brace, "}", tells us when the list of names of members of the set stops. Note that it does not make any difference in which order we list the members of a set. {3, 4, 7} and {7, 4, 3} and {4, 3, 7} are all the same sets.

We may also use this "set notation" for infinite sets. For example, we would write "{1, 2, 3, 4, ...}" to stand for the collection of all natural numbers. Here "..." stands for "and so forth". We shall return to a discussion of what the word "infinite" means above. You might look it up in a dictionary.

True or False -- as before:

4. (a) 7 is a member of {3, 10, 7}.
   (b) 4 is a member of {3, 10, 7}.
   (c) There are three members in {3, 10, 7}.
   (d) Every member of {3, 10, 7} is less than 32.
   (e) Every member of {5, 7, 19, 35} is less than 20.
   (f) Some member of {5, 7, 19, 35} is greater than 20.
   (g) Every member of {5, 10, 15, 20, 25} is greater than 4.
   (h) The largest member of {5, 7, 10} is 10.
   (i) The largest member of {5, 10, 7} is 10.
   (j) The smallest member of {5, 10, 15, 20, 25} is 5.
   (k) The smallest member of {3, 4, 5, 6, 7} is 3 and the largest member of this set is 7.

In mathematics (and other places), the largest member in a set is often called the maximum of the set. The smallest member of a set is called the minimum of the set. The maximum of {5, 10, 32} is 32 and the minimum of this set is 5.

Fill in the blanks:

5. (a) The maximum of $\{5, 25, 72, 10\}$ is ____.
   (b) The minimum of $\{4, 3, 7, 15\}$ is ____.
   (c) The minimum of $\{3, 4, 5, 6\}$ is ____.
   (d) The maximum of $\{3\}$ is ____.
   (e) An example of a set whose maximum is 4 and whose minimum is 2 is ____.
   (f) An example of a set whose minimum is 4 and whose maximum is 4 is ____.

Is it possible to find a set of natural numbers which has no minimum? That the answer is "No" to this question is intuitively clear -- or is it? In any event, the answer is "No", and the fact that this is so turns out to be a most deep and fundamental fact about the natural numbers.

Is it possible to find a set of natural numbers which has no maximum? Here the answer is "Yes". For consider the set $\{1, 2, 3, \ldots\}$ consisting of all the natural numbers. Is there a maximum in this set? There is not. We can prove this "by contradiction". Suppose there is a maximum natural number. Call it n. Then $n + 1$ is a natural number. But $n < n + 1$. So n is not a maximum after all! In other words, there is no largest natural number because if we are given any natural number n we can find a larger natural number, namely, $n + 1$.

As we look over the above results, we conclude that a (non-empty) set of natural numbers always has a minimum but may have no maximum.

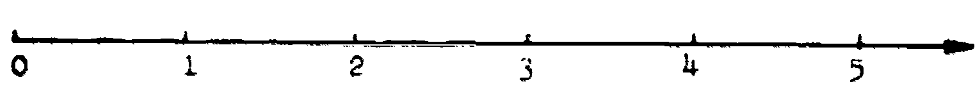
If you like fancy names for things (and like to sound sophisticated), you may express the fact that any non-empty set of natural numbers has a minimum element by saying that the collection of natural numbers is well ordered.

If someone asks what is the largest even natural number, the proper answer is, "There ain't no such animal." But it is more precise (and polite -- although somewhat stuffy) to say, "The set of even natural numbers has no maximum;" or "The maximum of the set of even natural numbers is undefined;" or "The maximum of the set of even natural numbers does not exist." Some of you may be tempted to say that the maximum of the set is infinity (whatever that means). But the maximum of a set must be in the set and whatever "infinity" is, it is not an even natural number. So "infinity" is certainly not the maximum of the set.

Fill in the blanks:

6. (a) The maximum of the set {30, 25, 20, 15, 10, 5} is _____.
   (b) The maximum of the set of all odd natural numbers is _____.
   (c) The largest natural number multiple of 3 is _____.
   (The multiples of 3 are 3, 6, 9, 12, 15, ...).
   (d) The smallest natural number multiple of 3 is _____.
   (e) An example of a set of natural numbers where every member is less than 3 and yet the set has no maximum is _____.
   (f) An example of a set of natural numbers with a minimum of 1 and a maximum of 1,000,000 is _____.
   (g) A number larger than 1,234,567,891,011,121,314,151,617,181,920,212,223 is ______________________________.
   (h) Which natural numbers are less than 1?
   (i) Which natural numbers are between 5 and 13?
   (j) Which natural numbers are between 27 and 23?
   (k) Which natural numbers are between 2 and 3?
   (ℓ) Which natural numbers are between 5976 and 5975?
   (m) If a is the minimum and b is the maximum of some set, then every member of the set is between a and b. (True or False?)
   (n) If a is the minimum and b is the maximum of some set, then every member of the set which is different from a and different from b is between a and b. (True or False?)
   (o) Which natural numbers are between 49 and 50?
   (p) If x is a natural number, there are no natural numbers between x and x + 1. (True or False?)
   (q) The minimum of the set of all natural numbers is _____.
   (r) The minimum of the set of all those natural numbers which are greater than 38 is _____.

One easy way to think about inequalities is in terms of a picture. Imagine a "number line" constructed as follows: On a horizontal line we choose a point called the origin and label it with the numeral "0". We label with the numeral "1" the point one unit to the right of the origin. We label with the numeral "2" the point two units to the right of the origin. We label with the numeral "3" the point 3 units to the right of the origin. And so on. The result looks like this,



In this way we may think of our natural numbers as identifying certain points on this line -- and vice versa. But, now we see that "is less than" (or "<") can be thought of as "to the left of". Also, "is greater than" (or ">") can be thought of as "to the right of". Also the meaning we gave to "between" agrees with our ideas about this picture. We could even think of the "minimum" of a set as the "leftmost member" of the set.

7. (a) What is the leftmost point among the points associated with the numbers in the set $\{17, 4, 3, 10\}$?
   (b) What is the rightmost point in the set $\{17, 4, 3, 10\}$?

Here we have started to be lazy again and instead of saying, "Rightmost point among the points associated with the numbers in the set", we said, "the points in the set". That is, we will do what most people do and talk about such a thing as "the point 10" instead of saying "the point on our line which was labeled '10'."

One-third of a pie is smaller than two-thirds of a pie. Half a loaf is better than none. Winning seven-ninths of the games you played this season is doing better than winning only two-ninths of them.

<u>True</u> <u>or</u> <u>False</u>:

8. (a) $\frac{1}{3} < \frac{2}{3}$
   (b) $\frac{2}{3} < 1$
   (c) $\frac{2}{3}$ is between $\frac{1}{3}$ and 1.
   (d) $\frac{1}{2} > 0$
   (e) $\frac{1}{2} < \frac{3}{2}$
   (f) 2 is between $\frac{1}{3}$ and $\frac{3}{2}$.
   (g) $\frac{2}{5}$ is between $\frac{1}{2}$ and $\frac{3}{2}$.
   (h) $\frac{1}{751} < \frac{2}{751}$
   (i) $\frac{43}{751} < \frac{34}{751}$
   (j) $.3 < .5$

Remember that ".3" is an abbreviation for "$\frac{3}{10}$", ".43" is an abbreviation for "$\frac{43}{100}$", ".759" is an abbreviation for "$\frac{759}{1000}$", etc.

9. (a) $.51 < .52$
   (b) $.76 < .19$
   (c) $.09 > .10$
   (d) $.4319 > .4098$
   (e) $.35 > .30$
   (f) $.51 > .5$
   (g) $.89 > .9$
   (h) $.413$ is between $.400$ and $.500$.
   (i) $.3$ is between $.24$ and $.29$.
   (j) The maximum of $\{\frac{3}{5}, \frac{2}{5}, \frac{1}{5}\}$ is _____.
   (k) The minimum of $\{.7, 1.6, 4.2\}$ is _____.
   (ℓ) The maximum of $\{.1, .2, .3, .4, .5, .6, .7, .8, .9, 1\}$ is _____.
   (m) Every member of $\{3, 5, \frac{7}{3}, 10, 1\}$ is greater than $\frac{1}{3}$.
   (n) Every member of $\{3, 5, \frac{7}{3}, 10, 1\}$ is less than 7.
   (o) Every member of $\{3, 5, \frac{7}{3}, 10, 1\}$ is less than 20.
   (p) Every member of $\{3, 5, \frac{7}{3}, 10, 1\}$ is less than 50.
   (q) Every member of $\{3, 5, \frac{7}{3}, 10, 1\}$ is greater than 1.
   (r) Every member of $\{3, 5, \frac{7}{3}, 10, 1\}$ is no greater than 10.
   (s) Every member of $\{7, 5, 9, 15, \frac{4}{3}, 95\}$ is at least as large as $\frac{4}{3}$.
   (t) No member of $\{33, 47, 53, 9.96\}$ is greater than 53.
   (u) No member of $\{5, 4, 3, 6, 9, 11\}$ is less than 4.

If every member of a set is at least as large as some number A, then we say that the number A is a <u>lower bound</u> for the set. Thus, 4 is a lower bound for $\{10, 5, 6.94, 32\}$ since every member of this set is at least as large as 4. Note that 3 is also a lower bound for this same set. Indeed, 5 is also.

If every member of a set is no greater than some number B, then we say that B is an <u>upper bound</u> for the set. Thus, 17 is an upper bound for $\{4, 3.6, 9, 11, 7\}$. 15 is also an upper bound for this same set. Indeed, 11 is also.

True or false or fill in the blank:

10. (a) 100 is an upper bound for $\{\frac{3}{2}, 4, 10, 94\}$.

(b) 10,479 is an upper bound for $\{\frac{3}{2}, 4, 10, 94\}$.

(c) 100 is a lower bound for $\{100, 1000, 10000\}$.

(d) 1 is a lower bound for $\{100, 1000, 10000\}$.

(e) $\frac{1}{2}$ is a lower bound for $\{100, 1000, 10000\}$.

(f) The set of all natural numbers which are upper bounds for $\{5, 19, 36\}$ is $\{36, 37, 38, 39, \ldots\}$.

(g) The set of all natural numbers which are lower bounds for $\{5, 19, 36\}$ is _____.

(h) The set of all natural numbers which are lower bounds for $\{3, 5, 9, 66\}$ is _____.

(i) The set of all natural numbers which are upper bounds for $\{10, 100, 1000, 10000\}$ is _____.

(j) The set of all natural number which are upper bounds for $\{\frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{4}{9}, \frac{5}{9}, \frac{6}{9}, \frac{7}{9}, \frac{8}{9}, 1\}$ is _____.

(k) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$, and $\frac{6}{7}$ are all between $\frac{1}{4}$ and 1.

(ℓ) .301, .302, .303, .304, ..., .399 are all between .3 and .4.

(m) .34196 is between .3419 and .3420.

(n) $\frac{11}{17} < \frac{10}{11}$.

In this last problem we need to know how to compare the size of two numbers when they are written as fractions with different denominators. Here we make use of some basic facts about fractions and find ways to express the numbers so that they have the same denominator. In problem 10(n), for example, we can write $\frac{11}{17} = \frac{11 \cdot 11}{11 \cdot 17} = \frac{121}{187}$ and $\frac{10}{11} = \frac{10 \cdot 17}{11 \cdot 17} = \frac{170}{187}$. Then since $\frac{121}{187} < \frac{170}{187}$, we see that the statement in problem 10(n) is true. For another example, we see that $\frac{3}{4} > \frac{2}{3}$ since $\frac{3}{4} = \frac{9}{12}$ and $\frac{2}{3} = \frac{8}{12}$ and $\frac{9}{12} > \frac{8}{12}$.

True or False:

11. (a) $\frac{1}{2} < \frac{1}{3}$

(b) $\frac{3}{2} > \frac{5}{4}$

(c) $\frac{29}{30} < \frac{39}{40}$

(d) $\frac{43}{75} > \frac{37}{60}$

(e) $\frac{31}{19} < \frac{30}{18}$

(f) $\frac{40}{39} > \frac{17}{17}$

(g) $\frac{43}{43} < \frac{360}{180}$

(h) $.32 < \frac{1}{3}$

(i) $4.31 > \frac{41}{10}$

(j) $.999 < 1$

(k) $394 < \frac{1000}{3}$

(ℓ) $\frac{1}{17} < \frac{1}{18}$

(m) $\frac{1}{31} < \frac{1}{30}$

(n) If a pie is divided into 3 equal pieces, the pieces are larger than if the pie had been divided into 4 equal pieces.

(o) $\frac{1}{3} > \frac{1}{4}$

(p) If your rich uncle leaves you 1/10th of his estate in his will, you get more than if he had left you 1/9th of it.

(q) $\frac{1}{10} > \frac{1}{9}$

(r) $\frac{1}{6} > \frac{1}{7}$ and $\frac{1}{7} > \frac{1}{8}$ and $\frac{1}{8} > \frac{1}{9}$

(s) $\frac{1}{50}$ is between $\frac{1}{49}$ and $\frac{1}{51}$.

(t) If x is a natural number, then $\frac{1}{(x + 1)} < \frac{1}{x}$.

(u) $0 < \frac{1}{2}$

(v) $0 < \frac{1}{456}$

(w) If x is any natural number then $0 < \frac{1}{x}$.

As we discovered before, a picture often helps in dealing with inequalities. On the number line we constructed before, it is easy to think of certain points as corresponding o numbers of the kind we have been using. For example, if we find, on the number line, the point which divides the segment between 0 and 1 into two equal pieces, it is natural to label this point as "$\frac{1}{2}$". If we choose two points on the number line which divide the segment from 0 to 1 into three equal pieces it is natural to label these two points (from left to right) as "$\frac{1}{3}$" and "$\frac{2}{3}$".

In a similar fashion we may find the points corresponding to $\frac{1}{4}$, $\frac{2}{4} = \frac{1}{2}$ and $\frac{3}{4}$.

To find the point corresponding to $\frac{43}{17}$ we would first think of $\frac{43}{17}$ as $\frac{34}{17} + \frac{9}{17} = 2 + \frac{9}{17}$, then choose points dividing the segment from 2 to 3 into 17 equal pieces, and then label the 9th one of these as "$\frac{43}{17}$".

As we discovered with the natural numbers on the line, it is the case that "<" can still be thought of as "to the left of". Also ">" can still be thought of as "to the right of". This fact is perhaps now not so "obvious", but it is true.

In terms of our number line, to say that B is an upper bound for a set is to say that no member of the set is to the right of B. To say that a number A is a lower bound for a set is to say that no member of the set is to the left of A.

In the following problems indicate in a picture some of the points that belong to the set and state whether the assertion is true or false:

12. (a) 5 is an upper bound for $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$.

    (b) 1 is the maximum of $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$.

    (c) The smallest of all the numbers which are upper bounds for $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ is 1.

The minimum of the collection of all the upper bounds for a set is called the <u>least upper bound</u> for the set.

True or False: (You may find a sketch on the number line is helpful.)

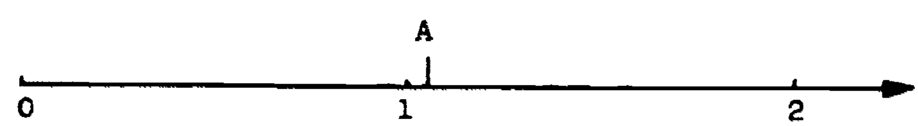
13. (a) The least upper bound for $\{1, 4, 5, 7\}$ is 6.
    (b) The least upper bound for $\{1, 4, 5, 7\}$ is 8.
    (c) The least upper bound for $\{1, 4, 5, 7\}$ is 7.01.
    (d) The least upper bound for $\{1, 4, 5, 7\}$ is 7.

Let us go into some detail on problem 13(c). How do we know that 7.01 is not the least upper bound for $\{1, 4, 5, 7\}$? It is certainly an upper bound for the set since 1, 4, 5, and 7 are all less than 7.01. But 7 is also an upper bound for the set. And $7 < 7.01$. Hence, 7.01 is not the least upper bound.

How are we certain of our answer in problem 13(d)? As we noted before, 7 is an upper bound for the set. Are we certain that 7 is the least upper bound? Yes. If a number B is less than 7, then B is not an upper bound for the set because 7 is in the set. We conclude that no upper bound for the set is less than 7 so that 7 must be the least upper bound.

More generally, we see in this fashion that if a set has a maximum, then that maximum is the least upper bound.

Consider the set $\{1.1, 1.01, 1.001, 1.0001, \ldots\}$. Is it clear what the set is meant to be? The member after 1.0001 is 1.00001. The member after that is 1.000001. And so on, without end. Is 1 a lower bound for this set? Surely it is. Is 1 the greatest lower bound for the set? In order to convince ourselves that it is, we would need to show that no number greater than 1 can be a lower bound for the set. Suppose A is a number greater than 1 as shown on the number line.



Isn't it clear that if we add to the interval between 1 and 2 the points that divide the interval into 10 equal pieces, then add the points that divide the interval into 100 equal pieces, then add the points that divide the interval into 1000 equal pieces, etc.; that we must eventually reach a stage where one of these points will fall between 1 and A? But this first point that does so must actually be in our set, for it will either be 1.1 or 1.01 or 1.001 or 1.0001 or, etc. Then we have a point in our set which is less than A. It follows that A is not a lower bound for the set. Since A was any number greater than 1, no number greater than 1 can be a lower bound for the set. We conclude that 1 is, indeed, the greatest lower bound.

True or False:

14. (a) 1 is a lower bound for $\{2.1, 2.01, 2.001, 2.0001, \ldots\}$.

(b) 2 is a lower bound for $\{2.1, 2.01, 2.001, 2.0001, \ldots\}$.

(c) 2 is the greatest lower bound for $\{2.1, 2.01, 2.001, 2.0001, \ldots\}$.

(d) 0 is a lower bound for $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$.

(e) 0 is the greatest lower bound for $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$.

(f) 0 is a lower bound for $\{3, \frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}, \frac{3}{6}, \frac{3}{7}, \ldots\}$.

(g) 0 is the greatest lower bound for $\{3, \frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}, \frac{1}{2}, \frac{3}{7}, \ldots\}$.

(h) 0 is the greatest lower bound for $\{\frac{1000}{1}, \frac{1000}{2}, \frac{1000}{3}, \frac{1000}{4}, \frac{1000}{5}, \ldots\}$.

Now we are beginning to get at some rather difficult ideas. Perhaps it would help us to review some of the basic ideas we have used. In doing so we shall also state some definitions more carefully than we have up to now. In stating these results, we sometimes make use of the phrase "if and only if". To assert that, "such and such, if and only if, so and so", is to say that if such and such is true then so is so and so, and, also, if so and so is true then so is such and such. In other words, "such and such, if and only if, so and so", is a short way of saying that such and such implies so and so, and, also, so and so implies such and such. If m, n, and p are natural numbers, $\frac{m}{p} < \frac{n}{p}$ if and only if $m < n$. If m, n, p, and r are natural numbers, $\frac{m}{p} < \frac{n}{r}$ if and only if $\frac{mr}{pr} < \frac{np}{pr}$. The largest element in a set is called the <u>maximum</u> of the set. The smallest element in a set is called the <u>minimum</u> of the set. A number A is a <u>lower bound</u> for a set if and only if no member of the set is less than A. A number B is an <u>upper bound</u> for a set if and only if no member of the set is greater than B. A number L is the <u>greatest lower bound</u> of a set if and only if L is the maximum of the collection of all the lower bounds for the set. A number U is the <u>least upper bound</u> of a set if and only if U is the minimum of the collection of all the upper bounds of the set.

The numbers we have been talking about here are 0 and numbers which can be written in the form $\frac{p}{q}$ where p and q are natural numbers. Such numbers are called <u>non-negative rationals</u>. Any number in the form $\frac{p}{q}$ where

p and q are natural numbers is said to be a positive rational number. As you may know, there are numbers which are said to be negative, but we will not consider these numbers in this study.

Since we have been writing names of some numbers in decimal fraction form, it would be well if we reminded ourselves that those numbers that we have mentioned in this way are positive rational numbers. For example:

$$2.0001 = \frac{20,001}{10,000} \text{ and } 13.4675 = \frac{134,675}{10,000} .$$

We have seen how zero or any positive rational number may be represented by a point on the number line. And we have seen further that "less than" for numbers may be thought of as "to the left of" for the corresponding points on the number line.

We discovered that any non-empty set of natural numbers must have a minimum, but some sets of natural numbers have no maximum (for example, the set of all natural numbers has no maximum).

Eventually we must come around to talking about "infinity" so let us ask a question that forces us to do so. ("Infinity" is fascinating anyway, so why put it off?) The question is, "Which sets of natural numbers don't have a maximum?" The answer is "Infinite sets of natural numbers." But what does that mean? It is easiest here to say what an infinite set is by saying what it is not. That is, we first define the phrase "finite set". And we do so in a very natural way. A finite set is a set with the property that it is possible to count the elements of the set. Thus, $\{3, 7, 1, 4, 75\}$ is a finite set because it is possible to count its elements (and, thus, discover that there are 5 elements in it). The set $\{2, 4, 6, 8, 10, \ldots, 1000\}$ is finite. It has 500 elements in it.

A set is said to be infinite if and only if it is not finite. Thus, the set $\{1, 2, 3, \ldots\}$ consisting of all natural numbers is infinite. The set $\{2, 4, 6, 8, 10, \ldots\}$ consisting of all even natural numbers is infinite.

Feel disappointed? To some extent you should. We have not defined "infinity". We have only defined the phrase "infinite set". Thus, "infinite" makes sense here only as an adjective modifying "set". We have not defined the noun "infinity" and will not need to. We shall get into enough trouble with just the adjective.

It is more or less obvious that every finite set of natural numbers has a maximum, while no infinite set of natural numbers has a maximum. (It would be a good exercise in the careful use of language to try to write up a "proof" of these two facts.)

It's been too long since *you* did some of the work. In each of the following, state whether the set is finite or infinite and if possible find its maximum and its minimum:

15. (a) $\{1, 3, 5, 7, 9, \ldots, 999, 1001\}$

(b) $\{1001, 999, 997, \ldots, 5, 3, 1\}$

(c) $\{3, 9, 12, 4, 1\}$

(d) $\{1, 3, 5, 7, 9, \ldots\}$

(e) $\{1000, 1001, 1002, 1003, \ldots\}$

(f) $\{1000, 999, 998, 997, \ldots, 1\}$

(g) $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\}$

(h) $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{305}\}$

(i) The collection of all the Presidents of the United States of America.

(j) The collection of all the people in your school.

(k) The collection of all the leaves on all the trees in your state.

(ℓ) The collection of all the molecules in or on the earth.

(m) $\{2.1, 2.01, 2.001, 2.0001, \ldots\}$

(n) $\{.1, .01, .001, .0001, \ldots\}$

(o) $\{2.3, 3.3, 4.3, 5.3, 6.3, \ldots\}$

(p) $\{.3, .33, .333, .3333, \ldots\}$

(q) $\{.1, .11, .111, .1111, .11111, \ldots\}$

Now perhaps you see some of the "trouble" we are in! Some of these are a little tricky to understand. For instance, in 15(g), how can we be certain that $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$ has no minimum? Well, if it had a minimum it would be $\frac{1}{n}$ for some natural number n, since only numbers in this form are in our set. But $\frac{1}{(n+1)} < \frac{1}{n}$ and $\frac{1}{(n+1)}$ is in our set. So $\frac{1}{n}$ is not the minimum after all! So we have here a set of numbers which does not have a minimum. (Of course, it is not a set of natural numbers since every non-empty set of natural numbers has a minimum.) If we wish to use technical language we can say that we have shown that the set of all positive rational numbers is <u>not well ordered</u>.

Also, we see that some infinite sets of rational numbers (such as $\{2.1, 2.01, 2.001, 2.0001, \ldots\}$) may have a maximum -- contrary to the situation when we consider sets of natural numbers. So the "order situation" for the positive rational numbers is strikingly different from that for the natural numbers.

Give an example of a set of positive rational numbers that has the stated property or else state that this is impossible:

15. (a) A set with 3 elements whose maximum is 6.
    (b) A set with 3 elements whose maximum is 1.
    (c) A set with 10 elements whose maximum is 1.
    (d) A set whose maximum is 7 which has an upper bound of 5.
    (e) A set with 5000 elements whose maximum is 5000.
    (f) A set with 5000 elements whose maximum is 1.
    (g) A set whose minimum is 4 and whose maximum is 5.
    (h) A set whose minimum is 7 and whose maximum is 6.
    (i) A set with 2 elements whose maximum is 3 and whose minimum is 2.
    (j) A set with 1000 members whose maximum is 1 and whose minimum is $\frac{1}{1000}$.
    (k) A set with a minimum of 4 and no maximum.
    (ℓ) A set with a maximum of 0.
    (m) A set with a maximum of 1 and no minimum.
    (n) A finite set with no minimum.
    (o) An infinite set with a maximum.
    (p) A set with neither a maximum nor a minimum.

Try these. True or False:

17. (a) 5 is an upper bound for $\{1.1, 2.1, 3.1, 4.1, \ldots\}$.
    (b) 5 is an upper bound for $\{\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \ldots\}$.
    (c) 5 is an upper bound for $\{5, 3, 2, 1, 4\}$.
    (d) 5 is an upper bound for $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$.
    (e) 5 is an upper bound for $\{4, 4.1, 4.11, 4.111, 4.1111, \ldots\}$.
    (f) 5 is an upper bound for $\{4.9, 4.99, 4.999, 4.9999, \ldots\}$.
    (g) Every set of natural numbers which has an upper bound is finite.
    (h) Every set of positive rational numbers which has an upper bound is finite.

(i) 3 is an upper bound for the collection of all the positive rational numbers between 2 and 4.

(j) 3 is an upper bound for the set of all the rational numbers between 1 and 2.

(k) The set of all rational numbers between 0 and 1 is finite.

(ℓ) The set of all natural numbers between 0 and 514,596,743 is finite.

(m) There is a rational number between 0 and 1.

(n) There is a rational number between $\frac{1}{2}$ and 1.

(o) There is a rational number between $\frac{3}{4}$ and 1.

(p) There is a rational number between $\frac{90}{91}$ and 1.

(q) If A is a positive rational number, then $\frac{A}{2} < A$.

(r) $\frac{1}{3} + \frac{3}{4} < \frac{2}{3} + \frac{5}{4}$

(s) $\frac{37}{19} + \frac{4}{12} < \frac{50}{19} + \frac{4}{12}$

(t) $\frac{37}{19} + \frac{4}{12} < \frac{50}{19} + \frac{5}{12}$

(u) $\frac{43}{19} + \frac{31}{6} < \frac{41}{19} + \frac{31}{6}$

(v) $\frac{13}{4} + \frac{3}{17} < \frac{53}{4} + \frac{104}{17}$

(w) $\frac{59}{101} + \frac{1}{2} < \frac{59}{101}$

(x) $\frac{334}{1000} + \frac{1}{39} > \frac{333}{1000}$

(y) $\frac{(5 + 6)}{2}$ is between 5 and 6.

(z) $\frac{(43 + 44)}{2} = 43 + \frac{1}{2}$

(aa) $\frac{(43 + 44)}{2}$ is between 43 and 44.

When we divide 10 by 3 by the usual process, we first get a "partial quotient" 3, then 3.3, then 3.33, then 3.333 -- but the division never "comes out even"! So, many books say "$\frac{10}{3} = 3.333\ldots$" where the expression on the right of the equality is called an "infinite decimal". But what in the world does "3.333..." mean? Are we supposed to write "3333..." over "10000..." for a positive rational number which is equal to it? Surely that

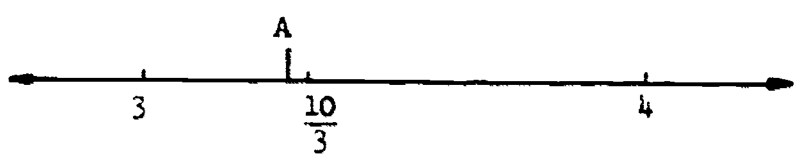
would be nonsense! Let's see if we can make some real sense out of "3.333...". Before we do, however, let us introduce a handy notation. In place of "3.3333..." we shall write "$3.\overline{3}$". That is, a bar over a string of decimals indicates that the string is to be repeated infinitely often; "$4.7\overline{56}$" means "4.756565656...". This notation will save us considerable writing and perhaps some confusion.

One more remark about the point of view we are adopting here and in what follows. We are assuming familiarity with the non-negative rational numbers -- and only with these numbers. Our major theme is to develop a good understanding of numbers that are <u>not</u> rational, but we have not yet introduced such numbers, and do not use them now.

Look at the collection $\{3, 3.3, 3.33, 3.333, \ldots\}$. We know what each of the members of this set is. What is $3.\overline{3} = 3.333\ldots$? How about thinking of it as the <u>least upper bound</u> of the above set?

Let's "prove" that $\frac{10}{3}$ <u>is</u> the least upper bound of the set $\{3, 3.3, 3.33, 3.333, \ldots\}$. $3.33333 = \frac{333,333}{100,000} = \frac{999,999}{300,000} < \frac{1,000,000}{300,000} = \frac{10}{3}$. Also, $3.3333333 = \frac{33,333,333}{10,000,000} = \frac{99,999,999}{30,000,000} < \frac{100,000,000}{30,000,000} = \frac{10}{3}$. Such calculations convince us that $\frac{10}{3}$ is an upper bound for our set. But is $\frac{10}{3}$ the <u>least</u> upper bound for the set?

Suppose A is a number such that $A < \frac{10}{3}$. We resort to a picture



Suppose we divide the interval from 3 to 4 into 3 equal pieces, then into 30 equal pieces, then into 300 equal pieces, then into 3000 equal pieces, etc. It is clear that eventually one of these division points will fall between A and $\frac{10}{3}$. But when this happens, the last point to the right of A which is to the left of $\frac{10}{3}$ will be in the form 99...9 over 300...0. That is, the number in decimal form will be 3.33...3 for some (finite) string of "3's". But such a number is in our set and exceeds A. Therefore, A is <u>not</u> an upper bound for the set. We have shown that no number less than $\frac{10}{3}$ is an upper bound for the set. We have also shown that $\frac{10}{3}$ is an upper bound for the set. We conclude that $\frac{10}{3}$ is the least upper bound of the set.

This suggests that we will not be led astry if we __define__ $3.\overline{3}$ to be the least upper bound of $\{3, 3.3, 3.33, 3.333, \ldots\}$.

In a similar vein, $.\overline{1}$ means the least upper bound of $\{.1, .11, .111, .1111, \ldots\}$, $.\overline{14}$ means the least upper bound of $\{.14, .1414, .141414, .14141414, \ldots\}$, and $.\overline{142857}$ means the least upper bound of $\{.142857, .142857142857, 142857142857142857, \ldots\}$.

Of course, these last three least upper bounds have simpler names. They are $\frac{1}{9}$, $\frac{14}{99}$, and $\frac{1}{7}$, respectively. (Carry out the divisions to convince yourself that this is reasonable in terms of the division process.)

__True__ or __False__:

18. (a) $\frac{1}{5}$ is the least upper bound of $\{.2, .22, .222, \ldots\}$.

(b) 1 is the least upper bound of $\{.8, .88, .888, \ldots\}$.

(c) 7 is the greatest lower bound of $\{7, 7.7, 7.77, 7.777, \ldots\}$.

(d) $\frac{2}{9}$ is the least upper bound of $\{.2, .22, .222, \ldots\}$.

(e) $\frac{2}{9} = 0.222\ldots$

(f) $\frac{4}{5}$ is the least upper bound of $\{.8, .80, .800, .8000, \ldots\}$.

(g) $\frac{4}{5} = .8000\ldots$

(h) $\frac{1}{8}$ is the least upper bound of $\{.12, .1212, .121212, \ldots\}$.

(i) 15 is the least upper bound of $\{1.2, 12.12, 121.212, 1212.1212, \ldots\}$.

(j) $.\overline{4}$ is the least upper bound of $\{.4, .44, .444, .4444, \ldots\}$.

(k) $.\overline{39}$ is the least upper bound of $\{3, .39, .3939, .393939, \ldots\}$.

(ℓ) $.\overline{39}$ is the least upper bound of $\{.39, .3939, .393939, \ldots\}$.

(m) $.999\ldots$ is the least upper bound for $\{.9, .99, .999, \ldots\}$.

(n) 1 is the least upper bound for $\{.9, .99, .999, \ldots\}$.

(o) $1 = .\overline{9}$

(p) $.4\overline{9}$ is the least upper bound for $\{.49, .499, .4999, \ldots\}$.

(q) $\frac{1}{2}$ is the least upper bound of $\{.49, .499, .4999, \ldots\}$.

(r) $\frac{1}{2}$ is $.4\overline{9}$

(s) $.9 = .8\overline{9}$

(t) $.9 = .89\overline{9}$

(u) $.\overline{3} + .\overline{6} = 1$

If you really understood parts (m), (n), and (o) above, you are making great strides. One of the things we must admit if we want to use infinite decimals is that some old friends now have some strange names. "$.\overline{9}$" <u>is</u> another name for 1. "$.4\overline{9}$" <u>is</u> another name for $\frac{1}{2}$. And so on. There is no other choice if we wish to preserve our ideas about the order of the numbers and their geometric representation on the line.

There is a point involved in the argument for 18(o) being true that we slid over in a slippery way before. How do we know that a set can't have <u>two different</u> least upper bounds. It <u>is</u> true that $.\overline{9}$ is a least upper bound of $\{.9, .99, .999, \ldots\}$, (by the definition of $.\overline{9}$) and also 1 is a least upper bound of $\{.9, .99, .999, \ldots\}$ (by an argument similar to those we have given before). But maybe $.\overline{9}$ is still different from 1. We shall show that this is not so, by showing that any set can have <u>at most one</u> least upper bound.

For suppose both a and b are least upper bounds of some set S. Then, of course, a and b are both upper bounds for S. If $a \neq b$, then either $a < b$ or $b < a$. Suppose $a < b$. Then, since a is an upper bound for S, b is not a least upper bound -- a contradiction. But, in a similar way, the supposition that $b < a$ implies that a is not a least upper bound -- a contradiction. We conclude that we must have $a = b$. In other words, there is actually only <u>one</u> least upper bound, if any.

We have opened the door for some important advances. But before we proceed, fill in the blanks in these basic definitions.

19. (a) A number B is an upper bound for a non-empty set S if and only if __________ member of S is greater than B.

    (b) A number A is a lower bound for a non-empty set S if and only if no member of S is __________ than A.

    (c) A number U is the least upper bound for a set S if U is the __________ of the collection of all upper bounds for S.

(d) A number L is the greatest lower bound for a set S if L is the maximum of the collection of ________.

(e) $.\overline{3}$ is the _____ _____ _____ of the set $\{.3, .33, .333, \ldots\}$.

(f) $.\overline{24}$ is the least upper bound of the set ________.

We have seen that some infinite decimals can be thought of as least upper bounds for certain sets of rationals. The infinite decimals with which we have dealt have been <u>repeating decimals</u>. That is, in each of them a certain block of digits was repeated over and over again to get the infinite decimal. Does every positive rational number have such an infinite decimal representation? The answer is yes.

The reason for this is clear although a detailed proof of it is a bit difficult to write down. Suppose $x = \frac{Q}{R}$ where Q and R are natural numbers. Start carrying through the process of dividing Q and R using the usual procedure in decimal notation. At each step in the division, the sub-remainder that is "brought down" must be less than R. After enough of the usual division steps have been carried through the non-zero digits in Q will be exhausted and we will be bringing down a zero each time. But after this happens we are bound to get a repetition of some exact situation we had before since there are at most R different sub-remainders. Thus, we must in this way generate a repeating decimal. (Of course, if the division comes out even the "repeating part" will be an infinite string of "0's"). Now it can be shown (and it actually does require proof) that the infinite decimal generated in this manner is indeed equal to x. Since x was any positive rational number, we conclude that any positive rational number is either a terminating or a repeating decimal.

What about the converse question? Are there perhaps repeating decimals which are not rational numbers? The answer is no. That this is so is most easily proven in terms of the language of infinite geometric series. We shall not give the proof here. Besides, what we are about to do makes the result seem most reasonable. We shall now show how to find (as a ratio of two natural numbers) the rational number named by an infinite decimal.

The method that we are about to use does not, by itself, actually prove that every repeating decimal names a rational number. This is because in our calculations we must do some arithmetic operations with infinite decimals as if they were rational numbers. That is, we must already assume they represent rational numbers to carry out our calculations. Consider the following equalities:

2

$$(100)(.\overline{37}) = 37.\overline{37}$$
$$37.\overline{37} - .\overline{37} = 37$$
$$(10)(.5\overline{124}) = 5.\overline{124}$$
$$(10000)(.5\overline{124}) = 5124.\overline{124}$$
$$(10000)(.5\overline{124}) - (10)(.5\overline{124}) = 5119$$

It should be clear that these equalities <u>do</u> need justification. After all, infinite decimals are different from finite decimals and how much of what you can do with finite decimals carries over to infinite decimals is really <u>not</u> obvious. (For example, how do you carry over the usual process for multiplying finite decimals to the case of infinite decimals? What is $.\overline{42}$ multiplied by $.\overline{5}$?)

We shall not give detailed arguments to justify the equalities displayed above. They <u>do</u> hold. In general, you may get the result of multiplying an infinite decimal by a power of ten by moving the decimal point appropriately. And you may add or subtract infinite decimals in the fashion you would expect.

Using these facts, if $x = .\overline{37}$, then $100x = 37.\overline{37}$ and, on subtracting, we get $99x = 37$. Then $x = \frac{37}{99}$ and we have expressed x as a ratio of natural numbers.

If $x = 15.3\overline{124}$, then $10x = 153.\overline{124}$, $10{,}000x = 152{,}124.\overline{124}$, and, on subtracting, $9{,}990x = 151{,}971$. Hence, $x = \frac{151{,}971}{9{,}990}$.

In this manner, we may express any repeating decimal as a ratio of natural numbers.

Our grand conclusion from all the above is that <u>a number is rational if and only if it has a terminating or repeating decimal expansion</u>.

But this is a strange kind of conclusion since the only numbers we have considered so far <u>are</u> rational numbers. At this stage, if a number isn't rational it simply <u>isn't</u>.

On the other hand, we do have infinite decimals that are not repeating. Examples abound: .101001000100001..., .343444344444434444443..., .12345678910111213141516..., etc.

If these infinite decimals mean anything, they are certainly not names for rational numbers. We must invent a new kind of number for these symbols to name! The door is now open to do this, but first do the following problems:

True or False:

20. (a) .001001001001... is a rational number.
    (b) .110110110110... is a rational number.
    (c) .767667666766667... is a rational number.
    (d) .799999... is a rational number.
    (e) .699999... = .7
    (f) .74747474... is the least upper bound of {.74, .7474, .747474, ...}.
    (g) .69999... is the least upper bound of {7, 6, 6.1, 6.01, 6.001, 6.0001, 6.00001, ...}.
    (h) The least upper bound of {.3, .33, .333, .3333, ...} is the same as the least upper bound of $\{\frac{1}{5}, \frac{1}{4}, \frac{1}{3}\}$.
    (i) $.14141414... = \frac{14}{99}$
    (j) $3.173737373... = \frac{3142}{990}$
    (k) $5.4319191919... = \frac{53776}{9900}$

The set {1, 1.01, 1.01001, 1.010010001, 1.01001000100001, ...} is a set of rational numbers. 2 is an upper bound for this set. In fact, all the numbers in the set are between .9 and 1.1. What is the least upp bound of this set which is bounded above? In a sense, this is the question that this whole presentation has been headed towards. There is an obvious answer to the question in terms of the machinery we have built up. The least upper bound should be the number named by the infinite decimal "1.01001000100001...". But this expression does not name a rational number. So, what does it mean? It is a name for the new kind of number we now <u>invent</u> to be exactly the thing we want -- the least upper bound of the set (of rationals) {1, 1.01, 1.01001, 1.010010001, ...}.

The temptation, of course, is simply to <u>define</u> 1.1010010001... to be the least upper bound of the set {1, 1.1, 1.101, 1.101001, ...}. But this won't work by itself. We got away with defining .3333... to be the least upper bound of {.3, .33, .333, .3333, ...} because it turned out that this least upper bound did exist among the rationals so that ".3333..." ended up being a name for something with which we were already familiar -- namely, $\frac{1}{3}$. In order to attach a meaning to "1.1010010001..." we must end up saying something like "1.1010010001... <u>is</u> defined to be such and such" where the "such and such" is something with which we were already familiar -- something already well defined. One thing to do is to define such a number to be a whole mess of sets of rational numbers -- sets of rationals which (on an

intuitive basis) "have the given number as least upper bound." This is what we shall do. In order to do so we shall bring in a new idea.

If A and B are sets of rational numbers, we shall say that A and B <u>get equally large</u> if either

(i) A and B have the <u>same</u> rational least upper bound or

(ii) given any a in A, there is a b in B such that $b \geq a$, and, vice versa, given any b in B, there is an a in A such that $a \geq b$.

For example, $\{5, 6\}$ and $\{4, 6\}$ get equally large while $\{5, 7\}$ and $\{4, 6\}$ do not.

$\{1\}$ and $\{.9, .99, .999, .9999, \ldots\}$ get equally large since they both have the same rational least upper bound.

$\{1, 2, 3, 4, 5, \ldots\}$ and $\{2, 4, 6, 8, \ldots\}$ clearly get equally large.

$\{.1, .101, .101001, .1010010001, \ldots\}$ and $\{.101, .1010010001, .101001000100001000001, \ldots\}$ get equally large.

For each of the following, state whether or not the two sets get equally large. ("yes" or "no" will do.)

21. (a) $\{4.7, 10\}$, $\{10, 7, 4\}$

(b) $\{10, 7, 4\}$, $\{10, 11, 12\}$

(c) $\{7, 7.1, 7.11, 7.111, \ldots\}$, $\{4, 10, 7\}$

(d) $\{7, 7.1, 7.11, 7.111, \ldots\}$, $\{6, 7, \frac{64}{9}\}$

(e) $\{3, 4, 5, 6, 7, \ldots\}$, $\{1, 4, 9, 16, 25, \ldots\}$

(f) $\{1, 2, 3, 4, 5, \ldots\}$, $\{100, 100.1, 100.11, 100.111, \ldots\}$

(g) $\{.1, .11, .111, \ldots\}$, $\{.1, .109, .1099, .10999, .109999, \ldots\}$

(h) $\{.1, .11, .111, .1111, \ldots\}$, $\{.1, .101, \frac{2}{18}\}$

(i) $\{.1, .101, .1010010001, .101001000100001, \ldots\}$, $\{.1, \frac{1}{20}, .101, \frac{1}{30}, .101001, \frac{1}{40}, .1010010001, \frac{1}{50}, .101001000100001, \frac{1}{60}, \ldots\}$

(j) $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots\}$, $\{1\}$

(k) {.2, .1, .202, .101, .202002, .101001, .2020020002, .1010010001, ...}, {.2, .202, .20202, .2020202, ...}

(ℓ) $\{1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \frac{1}{5}, 5, \ldots\}$, $\{53, \frac{53}{2}, \frac{53}{3}, \frac{53}{4}, \frac{53}{5}, \frac{53}{6}, \ldots\}$

(m) $\{.\overline{113}\}$, {.113, .113113, .113113113, ...}

(n) {.1, .101, .10101, .1010101, ...}, {1, .10, .101, .1010, .10101, .101010, ...}

(o) {.23, .2323, .232323, ...}, {.2, .23, .232, .2323, .23232, ...}

We are going to use sets of rationals to identify our new numbers. We want two sets of rationals to identify the same number if they get equally large. Of course, we are not interested in sets of rationals such as {1, 2, 3, ...} which are unbounded on the right, so we restrict our attention to sets of rationals which are bounded above. So, finally, we may state our basic definition. A <u>real number</u> is a collection of sets of rationals which are bounded above and all of which get equally large. Thus, we do not simply <u>associate</u> our new real numbers with certain sets of rationals. We actually <u>define</u> a real number <u>to be</u> a collection of sets of rationals.

We need a way to name these new objects we call real numbers. One is at hand. The collection of all those sets of rationals which get equally as large as $\{b, b.a_1, b.a_1a_2, b.a_1a_2a_3, \ldots\}$ where "b" is a decimal name for a natural number and the a's are decimal digits, is denoted by "$b.a_1a_2a_3\ldots$". So, with perhaps some feeling of accomplishment, we can now say exactly what any infinite decimal means. ".101001000100001..." stands for the collection of all sets of rationals which get equally as large as {.1, .10, .101, .1010, .10100, .101001, .1010010, ...}.

If this were to be a complete logical development of the real number system, we would now launch into the statement and proof of a series of theorems that follow from our definition. We would first need to define addition and multiplication for real numbers and tell what it means for one real number to be less than another. Much of this goes through in a very natural way when we use the infinite decimal names for the reals. One of the next things we would do would be to show that those real numbers which we naturally now call <u>rational real numbers</u> (i.e., those collections of sets of rationals which all have the same rational least upper bound) do behave as rational numbers should.

There has probably been too much rigor in these last few paragraphs, anyway. It's fairly certain you can answer the following questions even though we haven't actually stated all the needed definitions. (True or False)

22. (a) $.101001000100001\ldots + .010110111011110\ldots = .11111\ldots$

(b) $.010110111011110\ldots < .101001000100001\ldots$

(c) $5.431 < 5.4309009000900009\ldots$

(d) $(3)(.101001000100001\ldots) = .303003000300003\ldots$

(e) $.124 < .12345678910111213\ldots$

(f) $(7)(.\overline{12}) = .\overline{84}$

(g) $(7)(.\overline{43}) = 3.\overline{01}$

(h) $.\overline{77} + .\overline{14} = .\overline{91}$

(i) $.\overline{77} + .\overline{24} = 1.\overline{01}$

Now that we have the real numbers clearly defined, what have we gained? In what ways do the real numbers as a system differ from the rational numbers as a system? Algebraically, the two systems are very similar. Addition and multiplication always make sense and are commutative and associative in both systems. Multiplication distributes across addition in both systems. And division, except by zero, is always possible in either system.

But, of course, the reals make up a more comprehensive system than the rationals since each rational can be thought of as a real number. In terms of our development, this amounts to treating the rational number $\frac{P}{Q}$ as identical with the real number which is the collection of all sets of rationals which get equally large with the set $\{\frac{P}{Q}\}$.

The basic difference between the rationals and the reals is usually expressed by saying that the real number system is <u>complete</u> while the rational number system is not. To say that a number system is <u>complete</u> is to say that whenever a set of numbers from the system is non-empty and bounded above, then that set has a least upper bound in the system. More briefly, a system is said to be <u>complete</u> if every non-empty set which has an upper bound has a least upper bound.

The system of natural numbers is complete in this sense since each set of natural numbers which is bounded above has a maximum and that maximum will be the least upper bound.

Completeness is lost, however, in moving from the natural numbers to the rational numbers system. For example, the set of rationals $\{1, 1.01, 1.01001, 1.010010001, \ldots\}$ is bounded above but has no rational least upper bound. The least upper bound for the given set is the real number 1.01001000100001... which is not rational.

Real numbers which are not rational are said to be irrational.

Before discussing the proof of the completeness of the real number system, let us look at other consequences of this completeness.

The first has to do with roots of equations.

23. Find the positive rational x for which

(a) $2x = 4$ $x =$ ____

(b) $3x - 7 = 5$ $x =$ ____

(c) $7x - 4 = 4$ $x =$ ____

(d) $5x + 10 = 20$ $x =$ ____

(e) $x^2 = 4$ $x =$ ____

(f) $x^2 = 144$ $x =$ ____

(g) $x^2 = 36$ $x =$ ____

(h) $x^2 = 25$ $x =$ ____

(i) $x^2 = \frac{25}{36}$ $x =$ ____

(j) $x^2 = \frac{36}{25}$ $x =$ ____

(k) $x^2 = \frac{1}{4}$ $x =$ ____

(ℓ) $x^2 = \frac{25}{4}$ $x =$ ____

(m) $x^2 = 3600$ $x =$ ____

(n) $x^2 = 1$ $x =$ ____

(o) $x^2 = .04$ $x =$ ____

(p) $x^2 = .36$ $x =$ ____

(q) $x^2 = 625$ $x =$ ____

(r) $x + 1 = x$ $x =$ ____

(s) $x + 7 = 3$ $x =$ ____

Consider 23(r) more closely. How are we certain that there is no rational number x for which $x + 1 = x$? If there were such a number, we would have, on subtracting x from both sides of the equation, that $x + 1 - x = x - x$. That is, $1 = 0$. But this is impossible. We conclude that no such x could exist.

Suppose we had asked for the positive rational number x for which $x^2 = 2$. Just as is the case when we ask for the x for which $x + 1 = x$, there is no such positive rational x! We shall now prove this.

First, we need the fact that the square of an even number is even and the square of an odd number is odd. To say that a natural number is even is to say that it is a multiple of 2. Thus, if x is even, we have $x = 2k$ for some natural number k. But then $x^2 = (2k)^2 = 4k^2$. Since $4k^2$ is clearly an even number, x^2 is even when x is. If x is odd, then x must be one greater than some even number. That is, for some k we must have $x = 2k + 1$. But then $x^2 = (2k + 1)^2 = (2k + 1)(2k + 1) = (2k + 1)2k + (2k + 1) = 4k^2 + 2k + 2k + 1 = 4k^2 + 4k + 1$ which is one greater than an even number. Hence, x^2 is odd when x is.

Now suppose there is a positive rational number $x = \frac{P}{Q}$ such that $x^2 = (\frac{P}{Q})^2 = 2$. There is no loss in generality in assuming that P and Q are not both even since this will certainly be the case when x is expressed as a fraction in lowest terms. Since $(\frac{P}{Q})^2 = 2$ we have $\frac{P^2}{Q^2} = 2$ and $P^2 = 2Q^2$. Then P^2 is even. But this implies that P itself is even since if P is odd so is its square. Therefore, for some natural number k, we have $P = 2k$. Then from $P^2 = 2Q^2$ we get $(2k)^2 = 2Q^2$, $4k^2 = 2Q^2$, and, finally, $2k^2 = Q^2$. Hence, Q^2 is even and so is Q. But this is a contradiction since either P or Q must fail to be even. Since the assumption of the existence of a rational number whose square is 2 leads to a contradiction, there cannot be any such rational number.

If we had only rational numbers, we could not solve the equation $x^2 = 2$. In the positive reals, however, we can do so. The solution is denoted by $\sqrt{2}$, is called the positive square root of 2, and is the real number which is the collection of all those sets of rationals which get equally large with the set of all rational numbers whose squares are less than 2.

It can be shown that not only is $\sqrt{2}$ irrational, but so is $\sqrt{N}$ when N is any natural number which is not the square of a natural number. Thus, none of the numbers $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, ... would exist if we had only rational numbers to work with, but they all are perfectly well defined (irrational) real numbers. In fact, the system of non-negative real numbers is closed under the process of taking square roots. That is, every non-negative real number has a real square root. An even stronger statement can be made in that no matter what natural number k and positive real number b we are given there is always a real number x such that $x^k = b$. In other words, the non-negative reals are closed under taking $k^{\underline{th}}$ roots. This is certainly not true of the rational number system.

24. Classify each of the following as to whether it is a rational number, an irrational number or undefined (i.e., not a real number):

(a) $\sqrt{16}$

(b) $\sqrt{11}$

(c) .034034034034...

(d) $.\overline{010203}$

(e) $\sqrt{\frac{4}{9}}$

(f) The collection of all sets of rationals which get equally large with $\{\frac{7}{3}\}$

(g) The collection of all sets of rationals which get equally large with the set of all rationals whose squares are less than 3.

(h) The collection of all sets of rationals which get equally large with the set of all rationals whose squares are less than 81.

(i) The collection of all sets of rationals which get equally large with {1, 2, 3, 4, ...}.

(j) The collection of all sets of rationals which get equally large with {1, 1 - .01, 1 - .01001, 1 - .010010001, 1 - .01001000100001, ...}.

(k) The collection of all sets of rationals which get equally large with {.3, .1, .31, .11, .31311, .111, .313113111, .1111, .31311311131111, .11111, ...}.

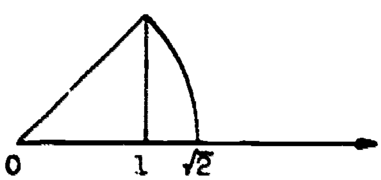
(ℓ) .1234567891011121314...

(m) The x for which $x^2 = -1$.

(n) The smallest positive real number.

(o) The least upper bound of the set of all reals whose squares are less than 2.

The move from the rationals to the reals is very important in terms of the number line. In the picture below we have shown the number line and a geometric construction of a length of $\sqrt{2}$.



The right triangle shown has both legs one unit long. By the Pythagorean Theorem, its hypotenuse must have length $\sqrt{2}$. What was shown in the last section tells us that if we had only rational points on the line, our arc would pass right through the line without hitting it. This, of course, should not happen for a line.

The completeness of the real number system can be expressed in geometric terms by saying that every point on the number line has a real coordinate. The converse is also true in that every real number corresponds to some point on the line.

According to the definition we gave for a real number, the real number b is identified with the collection of all sets of rationals which are not to the right of b and yet have members arbitrarily close to b.

This geometric picture will be helpful as we look at an incomplete sketch of the proof of the completeness of the reals. Let S be a set of real numbers which is bounded above by, say, b. Then for each x in S, $x \leq b$. Each x in S is a collection of sets of rationals. Clearly, none of the rationals in any of these sets could exceed b. Let A be the collection of all the rational numbers which belong to any of the sets corresponding to any x in S. Then A is bounded above by b. Let y be the real number consisting of the collection of all sets of rationals which get equally large with A. y will be the least upper bound for S.

This outline of the proof leaves many details to be filled in. You might try your hand at filling the gaps.

The word "real" we have used to identify the numbers we have introduced is, in a sense, unfortunate. Real numbers are neither more nor less real (in a non-technical sense) than are the rational numbers or other kinds of numbers. It is true that real numbers do seem to have a "concrete representation" on the number line. But it turns out that even so-called imaginary numbers also have a "concrete" geometric interpretation.

Indeed, the reals almost seem to be a bit "unreal" when we look at the care needed to build them from the rationals.

25. Correctly enter "Yes or "No" in each position in the following table:

The number system:	Natural numbers	Non-negative rationals	Non-negative reals
Every finite set of numbers has a maximum.			
Every non-empty set of numbers has a minimum.			
Every set of numbers which is bounded above has a maximum.			
Every set of numbers which has a maximum is finite.			
Every infinite set of numbers has no maximum.			
Between any two distinct numbers there is another number.			
Each number corresponds to a point on the line.			
Each point on the line corresponds to a number.			
Every non-empty set of numbers which has a least upper bound. has a maximum.			
Every non-empty set of numbers which is bounded above has a least upper bound.			

ANSWERS

1. (a) T
   (b) F
   (c) T
   (d) T
   (e) F
   (f) F
   (g) T
   (h) F
   (i) T
   (j) T
   (k) T
   (ℓ) T
   (m) F
   (n) T
   (o) F

2. (a) F
   (b) T
   (c) T
   (d) T
   (e) T
   (f) T
   (g) F

3. (a) T
   (b) T
   (c) F
   (d) T
   (e) T
   (f) T
   (g) F
   (h) F

4. (a) T
   (b) F
   (c) T
   (d) T
   (e) F
   (f) T
   (g) T
   (h) T
   (i) T
   (j) T
   (k) T

5. (a) 75
   (b) 3
   (c) 3
   (d) 3
   (e) {2, 3, 4} et. al.
   (f) {4} only

6. (a) 30
   (b) Does not exist
   (c) Does not exist
   (d) 3
   (e) Does not exist
   (f) {1, 387, 1,000,000} et. al.
   (g) 1,234,567,891,011,121,314,
       151,617,181,920,212,224
   (h) none
   (i) 6, 7, 8, 9, 10, 11, 12
   (j) 24, 25, 26
   (k) none
   (ℓ) none
   (m) F
   (n) T
   (o) none
   (p) T
   (q) 1
   (r) 39

7. (a) 3
   (b) 11

8. (a) T
   (b) T
   (c) T
   (d) T
   (e) T
   (f) F
   (g) T
   (h) T
   (i) F
   (j) T

9. (a) T
   (b) F
   (c) F
   (d) T
   (e) T
   (f) T
   (g) F
   (h) T
   (i) F
   (j) $\frac{3}{5}$
   (k) 4.2
   (ℓ) 1
   (m) T
   (n) F
   (o) T
   (p) T
   (q) T
   (r) T
   (s) T
   (t) T
   (u) F

10. (a) T
    (b) T
    (c) T
    (d) T
    (e) T
    (f) T
    (g) {1, 2, 3, 4, 5}
    (h) {1, 2, 3}
    (i) {10000, 10001, 10002, ...}
    (j) {1, 2, 3, ...}
    (k) T
    (ℓ) T
    (m) T
    (n) T

11. (a) F
    (b) T
    (c) T
    (d) F
    (e) T
    (f) T
    (g) T
    (h) T
    (i) T
    (j) T
    (k) F
    (ℓ) F
    (m) T
    (n) T
    (o) T
    (p) F
    (q) F
    (r) T
    (s) T
    (t) T
    (u) T
    (v) T
    (w) T

12. (a) T
(b) T
(c) T

13. (a) F
(b) F
(c) F
(d) T

14. (a) T
(b) T
(c) T
(d) T
(e) T
(f) T
(g) T
(h) T

15. (a) Min 1, Max 1001 Finite
(b) Min 1, Max 1001 "
(c) Min 1, Max 12 "
(d) Min 1, No Max Infinite
(e) Min 1000, No Max Infinite
(f) Min 1, Max 1000 Finite
(g) No Min, Max $\frac{1}{2}$ Infinite
(h) Min $\frac{1}{305}$, Max $\frac{1}{2}$ Finite
(i) No Min or Max Finite
(j) " " " " "
(k) " " " " "
(ℓ) " " " " "
(m) No Min, Max 2.1 Infinite
(n) No Min, Max 0.1 Infinite
(o) Min 2.3, No Max Infinite
(p) Min 0.3, No Max Infinite
(q) Min 0.1, No Max Infinite

16. (a) $\{4, 5, 6\}$ et. al.
(b) $\{\frac{1}{3}, \frac{1}{2}, 1\}$ et. al.
(c) $\{\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \ldots\}$ et. al.
(d) Impossible
(e) $\{1, 2, 3, \ldots, 5000\}$ et. al.
(f) $\{\frac{1}{5000}, \frac{1}{4999}, \ldots, 1\}$ et. al.
(g) $\{4, 5\}$ et. al.
(h) Impossible
(i) $\{2, 3\}$ only
(j) $\{\frac{1}{1000}, \frac{1}{999}, \ldots, 1\}$ et. al.
(k) $\{4, 5, 6, \ldots\}$ et. al.
(ℓ) $\{0\}$ only
(m) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ et. al.
(n) Impossible
(o) $\{2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$ et. al.
(p) $\{1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \ldots\}$ et. al.

17. (a) F
(b) F
(c) T
(d) T
(e) T
(f) T
(g) T
(h) F
(i) F
(j) T
(k) F
(ℓ) T
(m) T
(n) T
(o) T
(p) T

18. (a) F
(b) F
(c) T
(d) T
(e) T
(f) T
(g) T
(h) F
(i) F
(j) F
(k) F
(ℓ) F
(m) T
(n) T
(o) T
(p) T
(q) T
(r) T
(s) F
(t) T
(u) T

19. (a) No
(b) less
(c) minimum
(d) lower bounds for S
(e) least upper tound
(f) {.24, .2424, .242424, ...}

20. (a) T
(b) T
(c) F
(d) T
(e) T
(f) T
(g) T
(h) T
(i) T
(j) T
(k) T

21. (a) Yes
(b) No
(c) No
(d) Yes
(e) Yes
(f) No
(g) No
(h) Yes
(i) Yes
(j) Yes
(k) No
(ℓ) No
(m) Yes
(n) Yes
(o) Yes

22. (a) T
(b) T
(c) F
(d) T
(e) F
(f) T
(g) F
(h) T
(i) F

23. (a) 2
(b) 4
(c) $\frac{8}{7}$
(d) 2
(e) 2
(f) 12
(g) 6
(h) 5
(i) $\frac{5}{6}$
(j) $\frac{6}{5}$
(k) $\frac{1}{2}$

(ℓ) $\frac{5}{2}$

(m) 60

(n) 1

(o) .2

(p) .6

(q) 25

(r) No solution

(s) No solution

24. (a) rational

(b) irrational

(c) rational

(d) rational

(e) rational

(f) rational

(g) irrational

(h) rational

(i) undefined

(j) rational

(k) irrational

(ℓ) irrational

(m) undefined

(n) undefined

(o) irrational